

Zadania zespolone

Zadanie 1.

Wykonaj podane działania:

$$\text{a) } (-2 + 3i) + (7 - 8i); \quad \text{b) } (4i - 3) - (1 + 10i); \quad \text{c) } (\sqrt{2} + i) \cdot (3 - \sqrt{3}i); \quad \text{d) } \frac{2-3i}{5+4i}$$

Rozwiązanie

$$\text{a) } (-2 + 3i) + (7 - 8i) = 5 - 5i$$

$$\text{b) } (4i - 3) - (1 + 10i) = -4 - 6i$$

$$\text{c) } (\sqrt{2} + i) \cdot (3 - \sqrt{3}i) = 3\sqrt{2} - \sqrt{6}i + 3i + \sqrt{3} = 3\sqrt{2} + \sqrt{3} + (3 - \sqrt{6})i$$

$$\text{d) } \frac{2-3i}{5+4i} = \frac{(2-3i)(5-4i)}{(5+4i)(5-4i)} = \frac{10-8i-15i-12}{25-20i+20i+16} = \frac{-2-23i}{41} = -\frac{2}{41} - \frac{23}{41}i$$

Zadanie 2.

Oblicz moduły podanych liczb zespolonych

$$\text{a) } 4i \quad \text{b) } 12i - 5 \quad \text{c) } \sqrt{7} + \sqrt{29}i \quad \text{d) } (\sqrt{5} - \sqrt{3}) + (\sqrt{5} + \sqrt{3})i \quad \text{e) } \sin \alpha + i \cos \alpha$$

Rozwiązanie

$$\text{a) } |4i| = 4$$

$$\text{b) } |12i - 5| = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{165}$$

$$\text{c) } |\sqrt{7} + \sqrt{29}i| = \sqrt{(\sqrt{7})^2 + (\sqrt{29})^2} = \sqrt{7 + 29} = \sqrt{36} = 6$$

$$\begin{aligned} \text{d) } |(\sqrt{5} - \sqrt{3}) + (\sqrt{5} + \sqrt{3})i| &= \sqrt{(\sqrt{5} - \sqrt{3})^2 + (\sqrt{5} + \sqrt{3})^2} = \\ &= \sqrt{5 - 2\sqrt{15} + 3 + 5 + 2\sqrt{15} + 3} = \sqrt{16} = 4 \end{aligned}$$

$$\text{e) } |\sin \alpha + i \cos \alpha| = \sqrt{\sin^2 \alpha + \cos^2 \alpha} = \sqrt{1} = 1$$

Zadanie 3.

Zapisz podane liczby w postaci trygonometrycznej

$$\text{a) } -\sqrt{5} \quad \text{b) } -6 + 6i \quad \text{c) } -2i \quad \text{d) } \sqrt{3} + i \quad \text{e) } \sin \alpha - i \cos \alpha \quad \text{f) } 1 - i \cot \alpha \quad \text{g) } 1 + \cos \alpha + i \sin \alpha$$

Rozwiązanie

$$\text{a) } -\sqrt{5} = \sqrt{5}(\cos \pi + i \sin \pi)$$

$$\text{b) } -6 + 6i = 6\sqrt{2} \left(\cos \frac{3}{4}\pi + i \sin \frac{3}{4}\pi \right)$$

$$\text{c) } -2i = 2 \left(\cos \frac{3}{2}\pi + i \sin \frac{3}{2}\pi \right)$$

$$\text{d) } \sqrt{3} + i = 2 \left(\cos \frac{1}{6}\pi + i \sin \frac{1}{6}\pi \right)$$

$$\text{e) } \sin \alpha - i \cos \alpha = \cos \left(\alpha - \frac{\pi}{2} \right) + i \sin \left(\alpha - \frac{\pi}{2} \right)$$

$$\text{f) } 1 - i \cot \alpha = \frac{1}{\cos \left(\alpha - \frac{\pi}{2} \right)} \left(\cos \left(\alpha - \frac{\pi}{2} \right) + i \sin \left(\alpha - \frac{\pi}{2} \right) \right) \text{ gdzie } \alpha \neq k\pi \text{ dla } k \text{ całkowitego}$$

$$\text{g) } 1 + \cos \alpha + i \sin \alpha = 2 \cos \left(\frac{\alpha}{2} \right) \left(\cos \left(\frac{\alpha}{2} \right) + i \sin \left(\frac{\alpha}{2} \right) \right)$$

Zadanie 4.

Oblicz wartość następujących wyrażeń. Wynik podaj w postaci algebraicznej:

$$\text{a) } (1 + i)^7; \quad \text{b) } (\sqrt{3} - i)^{32}; \quad \text{c) } (-2 + 2i)^8; \quad \text{d) } (\cos 33^\circ + i \cdot \sin 33^\circ)^{10}; \quad \text{e) } \left(\frac{1-i}{\sqrt{3}+i} \right)^6;$$

$$\text{f) } \left(-\cos \frac{\pi}{7} + i \cdot \sin \frac{\pi}{7} \right)^{14}$$

Rozwiązanie

$$\begin{aligned} \text{a) } (1 + i)^7 &= \left(\sqrt{2} \left(\cos \left(\frac{\pi}{4} \right) + i \sin \left(\frac{\pi}{4} \right) \right) \right)^7 = (\sqrt{2})^7 \left(\cos \left(7 \cdot \frac{\pi}{4} \right) + i \sin \left(7 \cdot \frac{\pi}{4} \right) \right) = \\ &= 8\sqrt{2} \left(\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right) = 8 - 8i \end{aligned}$$

$$\begin{aligned} \text{b) } (\sqrt{3} - i)^{32} &= \left(2 \left(\cos \frac{11}{6}\pi + i \sin \frac{11}{6}\pi \right) \right)^{32} = 2^{32} \left(\cos \left(32 \cdot \frac{11}{6}\pi \right) + i \sin \left(32 \cdot \frac{11}{6}\pi \right) \right) = \\ &= 4294967296 \left(\cos \left(\frac{176}{3}\pi \right) + i \sin \left(\frac{176}{3}\pi \right) \right) = 4294967296 \left(\cos \left(\frac{2}{3}\pi \right) + i \sin \left(\frac{2}{3}\pi \right) \right) = \\ &= 4294967296 \cdot \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) = -2147483648 + 2147483648\sqrt{3}i \end{aligned}$$

$$c) (-2 + 2i)^8 = \left(2\sqrt{2} \left(\cos\left(\frac{3}{4}\pi\right) + i \sin\left(\frac{3}{4}\pi\right)\right)\right)^8 = (2\sqrt{2})^8 =$$

$$\left(\cos\left(8 \cdot \frac{3}{4}\pi\right) + i \sin\left(8 \cdot \frac{3}{4}\pi\right)\right) = 4096(\cos(6\pi) + i \sin(6\pi)) = 4096(\cos 0 + i \sin 0) =$$

$$4096$$

$$d) (\cos 33^\circ + i \cdot \sin 33^\circ)^{10} = \cos(10 \cdot 33^\circ) + i \sin(10 \cdot 33^\circ) = \cos 330^\circ + i \sin 330^\circ =$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2}i$$

$$e) \left(\frac{1-i}{\sqrt{3}+i}\right)^6 = \left(\frac{\sqrt{2}(\cos(\frac{7}{4}\pi) + i \sin(\frac{7}{4}\pi))}{2(\cos(\frac{1}{6}\pi) + i \sin(\frac{1}{6}\pi))}\right)^6 = \frac{(\sqrt{2})^6(\cos(6 \cdot \frac{7}{4}\pi) + i \sin(6 \cdot \frac{7}{4}\pi))}{2^6(\cos(6 \cdot \frac{1}{6}\pi) + i \sin(6 \cdot \frac{1}{6}\pi))} = \frac{8(\cos(\frac{21}{2}\pi) + i \sin(\frac{21}{2}\pi))}{64(\cos \pi + i \sin \pi)} =$$

$$= \frac{\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}}{8 \cdot (-1)} = \frac{\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}}{-8} = -\frac{\sqrt{2}}{16} - \frac{\sqrt{2}}{16}i$$

$$f) \left(-\cos \frac{\pi}{7} + i \cdot \sin \frac{\pi}{7}\right)^{14} = \left(\cos\left(\frac{6}{7}\pi\right) + i \sin\left(\frac{6}{7}\pi\right)\right)^{14} = \cos\left(14 \cdot \frac{6}{7}\pi\right) + i \sin\left(14 \cdot \frac{6}{7}\pi\right) =$$

$$= \cos(12\pi) + i \sin(12\pi) = \cos 0 + i \sin 0 = 1$$

Zadanie 5.

Oblicz następujące pierwiastki:

a) $\sqrt{4i-3}$;

b) $\sqrt[3]{8}$;

c) $\sqrt{-2i}$;

d) $\sqrt[4]{-8 + 8\sqrt{3}i}$;

e) $\sqrt[6]{1}$

Rozwiązanie:

a) $z = 4i - 3 = -3 + 4i$

$$|z| = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$z = -3 + 4i = 5\left(-\frac{3}{5} + \frac{4}{5}i\right)$$

ale

$$z = 5(\cos \alpha + i \sin \alpha)$$

więc

$$\sqrt{z} = \sqrt{5}\left(\cos\left(\frac{1}{2}\alpha\right) + i \sin\left(\frac{1}{2}\alpha\right)\right)$$

Skorzystamy ze wzorów

$$\sin\left(\frac{1}{2}\alpha\right) = \mp \sqrt{\frac{1 - \cos \alpha}{2}}$$

I

$$\cos\left(\frac{1}{2}\alpha\right) = \mp \sqrt{\frac{1 + \cos \alpha}{2}}$$

Ponieważ

$$\cos \alpha = -0,6$$

a

$$\sin \alpha = 0,8$$

więc

$$\frac{\pi}{2} < \alpha < \pi$$

ale

$$0 < \frac{\alpha}{2} < \frac{\pi}{2}$$

Oznacza to, że

$$\cos\left(\frac{1}{2}\alpha\right) = \sqrt{\frac{1 + \cos \alpha}{2}} = \sqrt{\frac{1 - 0,6}{2}} = \sqrt{\frac{0,4}{2}} = \sqrt{0,2} = \sqrt{\frac{20}{100}} = \frac{1}{5}\sqrt{5} = \frac{\sqrt{5}}{5}$$

$$\sin\left(\frac{1}{2}\alpha\right) = \sqrt{\frac{1 - \cos \alpha}{2}} = \sqrt{\frac{1 + 0,6}{2}} = \sqrt{\frac{1,6}{2}} = \sqrt{0,8} = \sqrt{\frac{80}{100}} = \frac{2}{5}\sqrt{5} = \frac{2\sqrt{5}}{5}$$

$$\sqrt{z} = \sqrt{-3 + 4i} = \sqrt{5} \left(\cos\left(\frac{1}{2}\alpha\right) + i \sin\left(\frac{1}{2}\alpha\right) \right) = \sqrt{5} \left(\frac{\sqrt{5}}{5} + \frac{2\sqrt{5}}{5}i \right) = 1 + 2i$$

lub

$$\sqrt{-3 + 4i} = -1 - 2i$$

b) Są 3 pierwiastki. Jeden z nich to 2, czyli w postaci trygonometrycznej

$$2 = 2(\cos 0^\circ + i \sin 0^\circ)$$

ponieważ

$$360^\circ : 3 = 120^\circ$$

Pozostałe 2, to

$$2(\cos 120^\circ + i \sin 120^\circ) = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -1 + \sqrt{3}i$$

lub

$$2(\cos 240^\circ + i \sin 240^\circ) = 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -1 - \sqrt{3}i$$

c) $z = -2i$

$$|z| = 2$$

$$z = -2i = 2(0 - i)$$

Czyli

$$\cos \alpha = 0$$

a

$$\sin \alpha = -1$$

$$\alpha = 270^\circ$$

$$\begin{aligned}\sqrt{z} = \sqrt{-2i} &= \sqrt{2}\left(\cos \frac{270^\circ}{2} + i \sin \frac{270^\circ}{2}\right) = \sqrt{2}(\cos 135^\circ + i \sin 135^\circ) = \sqrt{2}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) \\ &= -1 + i\end{aligned}$$

lub

$$\sqrt{z} = \sqrt{-2i} = 1 - i$$

d) $z = -8 + 8\sqrt{3}i$

$$|z| = \sqrt{(-8)^2 + (8\sqrt{3})^2} = \sqrt{64 + 192} = \sqrt{256} = 16$$

$$z = -8 + 8\sqrt{3}i = 16 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right)$$

ale

$$z = 16(\cos \alpha + i \sin \alpha) = 16 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right)$$

czyli

$$\cos \alpha = -\frac{1}{2}$$

a

$$\sin \alpha = \frac{\sqrt{3}}{2}$$

więc

$$\alpha = 120^\circ$$

$$\begin{aligned} \sqrt[4]{z} &= \sqrt[4]{-8 + 8\sqrt{3}i} = \sqrt[4]{16(\cos 120^\circ + i \sin 120^\circ)} = \sqrt[4]{16}(\cos 30^\circ + i \sin 30^\circ) \\ &= 2 \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i \right) = \sqrt{3} + i \end{aligned}$$

Pozostałe 3 pierwiastki, to

$$z = 2(\cos 120^\circ + i \sin 120^\circ) = 2 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) = -1 + \sqrt{3}i$$

$$z = 2(\cos 210^\circ + i \sin 210^\circ) = 2 \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i \right) = -\sqrt{3} - i$$

$$z = 2(\cos 300^\circ + i \sin 300^\circ) = 2 \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) = 1 - \sqrt{3}i$$

e) Jednym z pierwiastków jest

$$z = 1$$

Pozostałe pierwiastki, to

$$z = \cos 60^\circ + i \sin 60^\circ = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$z = \cos 120^\circ + i \sin 120^\circ = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$z = \cos 180^\circ + i \sin 180^\circ = -1$$

$$z = \cos 240^\circ + i \sin 240^\circ = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$z = \cos 300^\circ + i \sin 300^\circ = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$